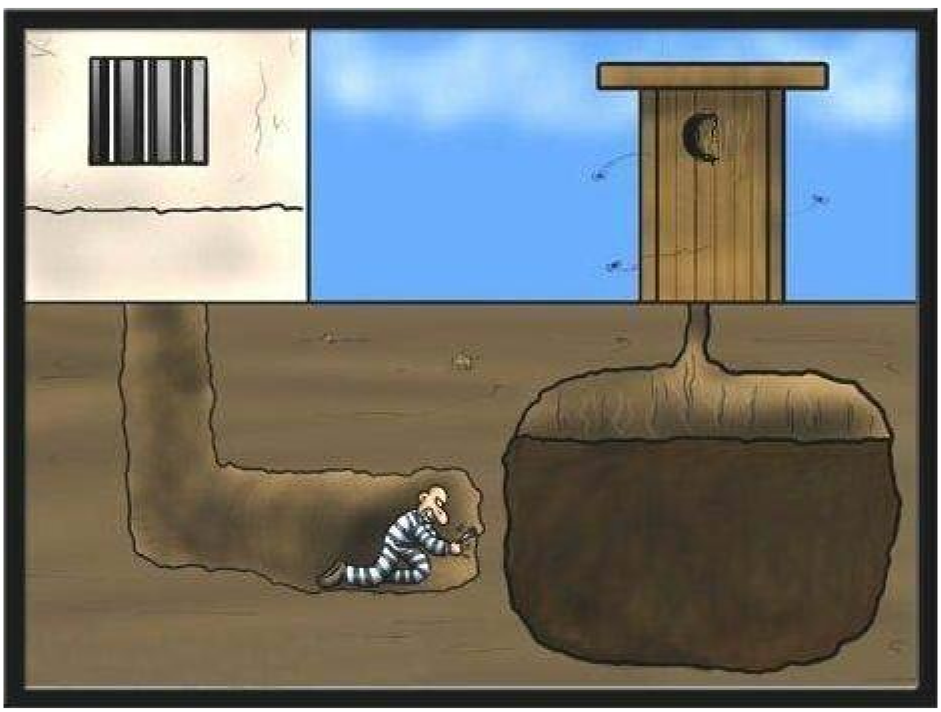
How Much Does The Number One Weigh?

And other methods for teaching arithmetic and algebra

by

Charles Carter Waid

A WORK IN PROGRESS



Acknowledgements

I make no claim of originality for the methods presented here. I am sure most, if not all, are in the literature, but most teachers I have worked with are unaware of their usefulness. In particular, using "numbers" to move things, rather than count or measure, as an educational tool is due to the great English geometer W.K. Clifford, appearing in his book "The Common Sense of the Exact Sciences". The arithmetic of the integers, real numbers, complex numbers and vectors become physically obvious using this paradigm. Five year olds can add positive and negative integers just as easily as they can positive integers. Bertrand Russell claimed this book, given to him on his fifteenth birthday, was the reason he became a mathematician. I think it is the most important book ever written for math educators (wait until you see what he does with geometry).

I would like to thank the children, the community, and the public schools of Lawton, Medicine Park and Elgin, Oklahoma, for letting me try these methods, create new ones and verify their effectiveness; the kindergarten and primary school children of the Baton Rouge International School for showing how easily the very young can understand negative numbers, variables and the language of algebra; and finally, the Katrina kids at FEMA's Renaissance Village in Baker, Louisiana, for proving that all children, even those who have endured enormous trauma and hardship, are natural born learners.

The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.
- Bertrand Russell

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Introduction

During my years of experience in teaching math, mostly at the college level, I found that getting students to learn math is primarily a psychological problem, not one of intellect. Confidence is the most important psychological factor: We all are aware of how math anxiety literally petrifies the minds of students. Several teaching experiments I have tried suggest that the confidence gained by becoming very good in one small part of math greatly improves learning in all the other areas.

During the last four years, my colleague, Thomas C. Cowsar, and I have been working with young students, he primarily with grades K through 6, I with grades 4 through 9. From this experience, I have developed some very strong opinions about the ability of young children to learn. I believe all pre- adolescent children free of serious physical, emotional, or environmental deficit are "gifted and talented" in our current use of the term, which is essentially that they can learn significantly more than they are currently being taught. They are, by nature, learning machines trying to understand the world; and they enjoy doing so. They are fundamentally realists, with minds uncluttered by simple-minded models of reality that so limit an adult's ability to learn. Kids' primary weakness is that they believe that adults, particularly their parents and teachers, are always right. Consequently, when they are told something that is either incorrect, or correct but explained incorrectly, they think, "That adult knows a lot more than me and therefore must be right. I don't understand them at all; therefore I must be dumb at math." I once had a teacher helping me in a math camp who had the kids convinced that $(-2)+(-3)=5$ because "two negatives make a positive" (simple-minded model). To counter this passive tendency I keep telling students:

> "If you have really thought about something, you should stick by your conclusions until presented with an actual, clearly presented counterexample. Quoting authority is not a counterexample!"

Five general approaches I have found to resonate with youngsters when teaching mathematics are:

- Let them discover what is going on by asking questions instead of giving answers.
- Emphasize the interplay between mathematics, language and grammar.
- Continually stress math as an efficient shorthand.
- Use parallel concepts as much as possible, from isomorphism to analogy to metaphor.
- Keep telling them that mathematics is not a spectator sport. You would never dream you could magically play the piano without practicing the scales and playing songs over and over.

Here are some quick examples that will be gone into in more detail later.

- How much does the number "one" weigh? What color is it?
- No, that is what it is! -- What does it mean?
- Numbers and numerical variables are nouns. The only difference is that the former are proper and the latter are common.
- Expressions are nouns. Equations and inequalities are sentences of the form noun/verb/noun where the nouns are expressions and verb is one of $=, \neq, <, \leq, >, \geq$.
- The meaning of the word "car" is apparently constructed by your brain from the set of all your experiences with cars, the sound <*car*>, pictures of cars, etc. Similarly, for the word green, green cars, etc. Green cars are all those things that are cars <u>and</u> green, i.e., the intersection of the set of car things and the set of green things. Set theory should be introduced in English classes!
- The relationship between exponents and multiplication is in exact parallel to the relationship between products and addition.
- The relationship between 0 and addition is in exact parallel (isomorphism) to that between 1 and multiplication.
- The relationship between negation and addition is in exact parallel to that between inversion and multiplication.

The methods given in this booklet were developed over the last two years while testing a board game that my wife, Margaret, also a mathematician, and I developed. The game, "Math Gym-1D"[1] is an implementation of Clifford's method of steps, which will be referred to occasionally throughout the text. I found serious gaps in many student's understanding of mathematics and developed a Power Point presentation of fifteen minute talks that I gave just before they started playing the game each day. The game gave them an enjoyable arena where they could practice what they had learned. This booklet expands these "Math Mini-Talks" into a more readable form and is intended as a resource for teachers.

1. Do Numbers Exist?

"How much does the number 'one' weigh?"

This is the first question I asked students of any age, followed with "What color is it?", "What does it sound like?", "Taste like?", "What is it made of ?" They usually discuss this for 10 minutes or so. You will probably have to point out to the students in the course of their discussion that the number one can be written with chalk on a blackboard, with pencil on a piece of paper, with a marking pen on a white board, etc. In addition, we could have expressed the number using the Roman numeral "I", or the print or cursive "one", or a simple tally mark "/". They will finally home in on the fact that the number

[1] "Math Gym-2D" is coming. It will give a physical realization to the arithmetic of complex numbers and two-dimensional vector algebra and geometry.

one does not physically exist; that it is a concept housed in our brain, presumably in the form of a pattern of firing of neurons. (If this is physical existence, then there is a different number 1 for each individual!) Then I ask them "By the way, how much does the word ‘the’ weigh?”

The purpose of this line of questioning is threefold. First to get them to understand that things like numbers, words, and many, many other things they deal with daily are **abstract concepts**, so I don't want to hear them complaining about mathematics being abstract. Secondly, I want them to realize that numbers are concepts that human beings made up to do useful things and we can make up new numbers as we please to do additional useful things. This sets the stage for using numbers as operators. Finally, I want to get them to start thinking about what things really are and what they really aren’t. This is essential for rational thought, a skill that our educational system is clearly failing to teach the general population.

At the end of class, I send them home to think about the following. If the number 1 only exists as a pattern of neural activity in your brain, then it is most likely learned from personal experience. That would mean that you have a different concept of 1 than I. So what is the number 1, really?

2. Symbols as Shorthand

One of the greatest causes of fear of mathematics is its use of symbolism. I have seen grown men and women grow pale when confronted with a variable symbols such as *x* or the summation sign symbol Σ. The purpose of this section is to get the students to realize that symbolism is used in all areas of endeavor, not just in mathematics, and they use it continually in their daily life.

Man uses symbols to quickly communicate concepts and complex information. Sounds such as words can be symbols, but we will stick with visual symbols. The letters of the alphabet, other then a or I, are symbols that in ordinary English do not contain information. Words, on the other hand, communicate complex concepts. The word “dog” brings to mind a visual image of a dog and questions as to its size, what breed it is, etc. Symbolism can be quite complex such as symbolism in the arts or psychology, however the symbolism of mathematics is much, much simpler. It is simply shorthand for normal English words or sentences. Mathematical symbolism is called mathematical **notation** and is said to be universal because the same symbols mean the same thing in any language. Math notation converts regular language into brief, easy to write expressions, much like I !U.

A brief history of the development of basic mathematical notation is provided in Appendix A. I am not suggesting that students be required to learn all this history. I include it primarily so that students can understand how long it took to develop our modern notational system, which in retrospect is very simple. I also think it could be a good source of Internet "lookup" projects. Other good projects would be to have students report on symbolism in other areas of human endeavor, such as sports, music, physics, chemistry, etc. and describe one or more notational systems they use. For example, in music, they can investigate classical treble and bass clef notation, guitar chord notation, or even that of shaped note hymns.

Math notation is not always appreciated!

3. Grammar and Math

Variables

I have found that a discussion of the grammar of mathematics is very helpful in overcoming students' fear of notation. In mathematical notation, 2 is a symbol for the number "two" while a letter of the alphabet such as *x* is shorthand for "a number"[2]. Both are names of things and are therefore called nouns in grammar. London and city are nouns too. Since the word “London” is the name of the specific town, it is called a

[2] Later on, depending upon the context, an alphabetic character may mean “a point", "a line", "a set", etc.

proper noun, while the word "city" does not name any specific town and is called a **common** noun. For exactly the same reason, 2 is a proper noun and the variable *x* is a common noun. **The only difference between a number and a variable is purely grammatical.** We use variables all the time in our daily life. We talk about cars, planes, and houses without giving it a second thought. Tell the students they should do the same thing when they see an *x*, *a*, α or ξ in a mathematical sentence.

> **All common nouns are variables. The word "variable" is a synonym for "common noun".**

In English, a common noun always refers to a certain set of proper nouns, e.g., an automobile always refers to a particular kind of powered, wheeled vehicle.
The set of actual things referred to by a common noun is called the **range** of the common noun. The game, "Math Gym-1D", introduces variables as "wild cards". Whenever a player gets a variable, say *x*, he immediately draws a card from the Variable Deck to find the range of the variable. He will see something like $x \in \{-3, 0, 2, 4\}$. These are the only values he can assign *x* when constructing his move.

Some common nouns are specializations of others. "Ford" and "convertible" are types of automobiles and their ranges are sub-sets of the range of "automobile". Exactly the same idea holds in mathematics. Start with the very general. "x is a number.", specialize to "x is a real number.", then to "x is a rational number", then to "x is an integer", and then "x is a natural number".

Expressions

The symbol $x+2$ means "a number plus two", or just "two more than a number", and $2\times x$ means "two times a number" or "twice a number".

$2+3$, 2×3 (or $2\cdot 3$), and $5+2\times 3$ are names of numbers also. They are proper nouns called <u>arithmetic expressions</u>.

$a+3$, $3\times y$ (or $3\cdot y$), and $a+3\times y$ and are names of numbers too. They are common nouns called <u>algebraic expressions</u>.

Evaluate (or calculate) an arithmetic expression means "Replace the expression with the standard name of the number it names."

Evaluate $2+3$. Ans. $2+3=5$ ($2+3$ is another name for 5)

Evaluate 2×3. Ans. $2\times 3=6$ (2×3 is another name for 6)

Evaluate an algebraic expression at certain values of its variables means "Substitute the values for the variables and evaluate the resulting expression."

Evaluate $\alpha+\beta$ when $\alpha=2,\ \beta=3.$

Ans. If $\alpha=2$ and $\beta=3,\ \alpha+\beta=2+3=5$

Evaluate $\alpha\times\beta$ when $\alpha=2,\ \beta=3.$

Ans. If $\alpha=2$ and $\beta=3,\ \alpha\times\beta=2\times3=6$

Of course, expressions can be complex:

$$2+3\times(5-2\times4)$$

$$1-(1-(1-(1-1)))$$

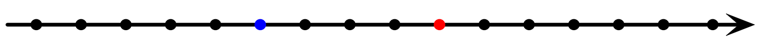
$$1+\cfrac{1}{1+\cfrac{1}{1+1}}$$

4. Numbers as Operators

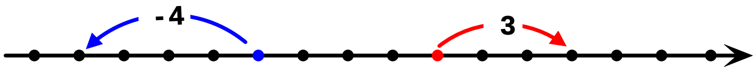
You have seen that numbers are concepts we have created to do useful things. You have used them to count and have used them to measure. Now we will show you how to use numbers to move things! The most important thing about movement is that it has both magnitude and direction. Two other important things are that it is easy to add movements --- you simply move by one and then by the other, and we can introduce the operation of negation --- just reverse direction.

Addition

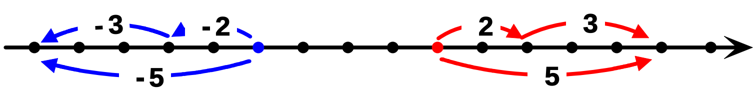
Suppose you have a line of dots as shown below and you were asked to move the red dot by 3 and the blue dot by –4. What would you do?



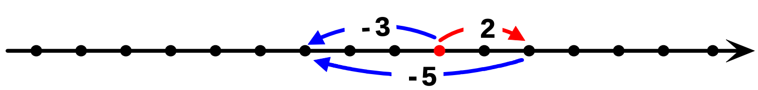
Most children will automatically answer as follows.



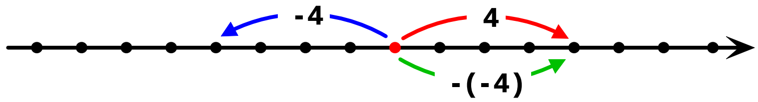
How would you add $2+3$ and $(-2)+(-3)$? Easy!



What about $2+(-5)$? Also easy!



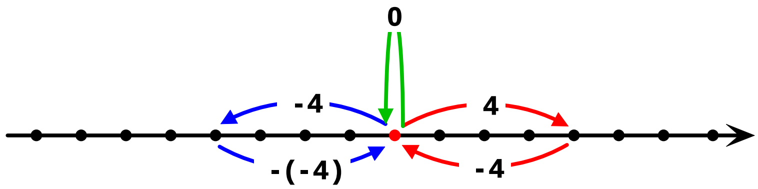
We now introduce the operation of **negation**. It is a **unary** operation because it operates only on one number. Addition, subtraction and multiplication are **binary** operations since they operate on two numbers. Negation of a move merely reverses the direction of the move.



It is this operation that allows us to move from arithmetic to algebra. First, we get the basic property of negation

The sum of any number and its negation is 0.

For example $4+(-4)=(-4)+(-(-4))=0$.



This property, along with the distributive law and zero's definition ($0+x=x$ for any number x) are all that are needed to prove that $0\cdot x=0$ for any number x.

Secondly, we can now properly define subtraction, as a combination of addition and inversion. To wit,

$$a-b\equiv a+(-b).$$

In other words, to subtract**, change the sign and add**.

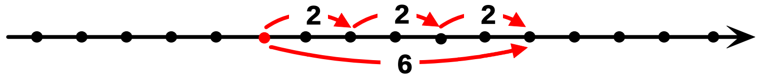
Definition: The **absolute value** of a number is its size or magnitude. It is computed by "dropping" the sign of the number.

Examples.

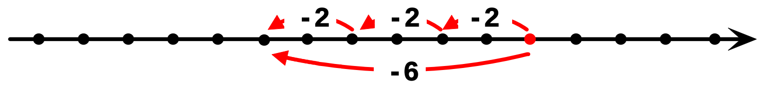
$$|2|=2$$
$$|-2|=2$$
$$|-0.0001|=0.0001$$
$$|56.132|=56.132$$

Multiplication

Multiplication by a positive integer is just repeated addition, so 2×3 is



Likewise $(-2)\times3$ is straightforward



Wait a minute! We have just seen that $(-2)\times 3$ without having memorized that the product of positive number with a negative number is negative!

Up to this point, the arithmetic of negative numbers has been physically obvious. We now come to the one point where the physical model (as we have been explaining it) fails. It is obvious that the product of a negative integer and a positive integer <u>in that order</u> is negative. What about the other way around? What about the product of two negatives? Well, we just have to define it:

> To multiply by a negative integer, multiply by its absolute value and reverse the direction of the product. Symbolically:
>
> If n is negative, $n\times a = -(|n|\times a) = |n|\times(-a)$.

This gives the students the most trouble, but they very quickly get used to it.

Examples.

$(-2)\times 3 = -(2\times 3) = 2\times(-3) = -6$

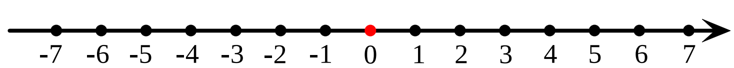
Notice that $(-2)\times 3 = 3\times(-2)$ (the commutative law holds)

$(-2)\times(-3) = -(2\times(-3)) = -(-6) = 6$

This anomaly has come up because I have been doing a little slight-of-hand to keep things simple. What is really going on is covered in the advanced section on Vectors, which I believe is just too tricky for early elementary school.

The Number Line and Order

You will have noticed that we have made no use of the concept of the number line. We need only the number line to help understand the ordering of the integers. It is constructed by picking any point on the line, called the **origin** (in red) and giving it the label (address) 0. Pick any other point on the line (usually to the right of 0) and give it the label 1. This defines the **unit of length**. Move out to the right in unit steps, labeling each point by the number of the steps from the origin. In the same way move out to the left from the origin and label each point by the negative of the number of steps from the origin.



Definition. If the number m is to the left of number n on the number line, then ***m* is less than *n*** ($m < n$) and ***n* is greater than *m*** ($n > m$).

$$\ldots -3 < -2 < -1 < 0 < 1 < 2 < 3 \ldots$$

Notice that if $m < n$, then $-m > -n$, i.e., **negating both numbers reverses the "sense" of the inequality**.

Summary

The game implements this by wrapping the line around into a closed loop and replacing the dots with spaces on which you move your playing token. However, there are several ways of implementing the method of steps in your classroom.

You can draw a "dotted" line on the black board and have you students work problems on it (or on their paper). For example ask them to "draw" the answer to $(-2)+1+(-3)=?$ or $2\times 2-7=?$ by moving the red dot and seeing how far and in what direction it has moved from its original position.

You can get your students to stand up and play a version of Simon Says, such as, "Simon Says walk two minus negative one"

If you have a tile floor you can use those for steps. If you are trying to get your students to learn ordered pairs and plane coordinates you can have them move

$(-2,1)$ (move left 2 & forward 1)

or

$(1,2)+(-1,-1)$ (move right 1 & forward 2, then left1 & backward 2)

I am sure you can think of other ways.

5. Algorithms for Integer Arithmetic

The preceding chapter shows why the arithmetic of the integers works the way it does. However, it does not give efficient methods for computing.

Multiplying

Forget about the rules that the product of two negative numbers is positive and the product of a negative number and a positive number is negative. Here are the rules you want to remember.

- The product of an **even** number of negative numbers is positive.

Examples.

$$(-2)\cdot(-3)=6$$

$$2\times(-2)\times 2\times(-2)=16$$

$$(-3)^4=81$$

- The product of an **odd** number of negative numbers is negative.

Examples.

$$(-1)\times(-2)\times(-3)=-6$$

$$(-1)\times(-2)\times(-3)\times(-1)\times(-2)=-12$$

$$(-2)^5=-32$$

The Multiplication Algorithm
When multiplying two or more numbers:

- Determine the sign via the odd or even rule. If it is negative, write down " – ".
- Multiply the absolute values of the number together. (Just ignore the negative signs and multiply

Examples.
$(-2)\times(-3)=2\times3=6$

$(-3)^4=3^4=81$

$(-1)\times(-2)\times(-3)\times(-1)\times(-2)=-(1\times2\times3\times1\times2)=-12$

$(-2)^5=-2^5=-32$

The Algorithm for Adding Two Numbers
When adding two numbers:

- The sign of the answer is that of the number having the larger magnitude. If it is negative, write down " – ".
- If they have the same signs, add the absolute values.
- If they have opposite signs, subtract the smaller magnitude from the larger.

Examples. $9+11=20$ (both positive)

$(-3)+(-17)=-(3+17)=-20$ (both negative)

$3+(-17)=-(17-3)=-14$ (opposite signs, $|-17|$ largest)

Algorithm for Adding Many Numbers
If you are adding more than two numbers, reduce it adding only two numbers

- Add all the positive numbers together.
- Add all the negative numbers together.
- Add the two results.

Example. $2-8+7-3=(2+7)+(-(3+8))=9+(-11)=-2$

Exercises.

$(-1)^3 \times (-2)^4 =$

$12 + (-34) =$

$29 - 14 =$

$12 + 4 =$

$(-2) + (-9) =$

$(-1) \times (-2) =$

$(-1) \times (-2) \times (-3) =$

$(-1) \times (-2) \times (-3) \times (-4) =$

$(-1) \times (-2) \times (-3) \times (-4) \times (-5) =$

$(-1)^{47} =$

$(-2)^6 =$

$2 \times (-3) \times (-5) \times 10 \times (-3) =$

$2 + 4 + (-5) + (-3) + 1 =$

$9 - 4 + 2 - 1 + 4 - 3 =$

$2 - 5 - 7 + 3 - 4 - 6 =$

$12.7 + 1.3 - 15.6 + 5.0 - 3.4 =$

6. Exponents (Powers)

Other than spelling and saying things like "exponentiation", exponents are not that difficult for students to learn once they see <u>why</u> they should be used. I always introduce the idea by putting

$$\underbrace{2 + 2 + \cdots + 2}_{97\ times} =$$

on the blackboard, have the students get out a clean sheet of paper and tell them to start writing 2 + 2 + 2, etc. until they had 97 of them. They moan and groan but start writing. Some of the smart ones will write 97×2 down, but I tell them, "That is not what I asked you to do. Write it out!" I let them go on for 30 seconds or so and then stop them. I ask them

> "Are you getting tired?"
> "Bored?"
> "Do you know how many you have already written down?"
> "When you finish will you really have 97? Maybe it will be 95 or 98!"
> and finally
> "Notice that when you are done writing, you haven't even started adding yet."

Then I tell them that mathematicians also got bored with doing this, so they made up shorthand for repeated addition called **multiplication**.

$$\underbrace{2+2+\cdots+2}_{97\ times}=97\times 2$$

I then put

$$\underbrace{2\times 2\times\cdots\times 2}_{97\ times}=$$

on the blackboard and point out that we have the same problem with repeated multiplication create yet another shorthand called

$$\underbrace{2\times 2\times\cdots\times 2}_{97\ times}=2^{97}$$

I tell them that 2 is called the **base** and 97 is called the **exponent** or **power** and point out that computing exponents is very difficult for all but small powers.

$$\underbrace{2+2+\cdots+2}_{97\ times}=97\times 2=194$$

$$\underbrace{2\times 2\times\cdots\times 2}_{97\ times}=2^{97}\approx 158,\underbrace{000,...,000}_{27\ zeroes}$$

Definition $n\times b$ means the sum of $n\,b$'s

(97×2 is the sum of 97 two's)

b^n means the product of n b's

(2^{97} is the product of 97 two's)

Every time students calculate an exponent wrong, say $2^3=6$, make them repeat the mantra “Two cubed is the **product** of three twos. Three times two is the **sum** of three twos. It seems to take a few days to a week to sink in, but this mantra really seems to help. Have them work many easy problems in pairs: ($a\times b$, a^b).

What about 0?

$$0\times 1=0\times 2=0\times 3=0\times 4=\cdots=0$$

$$1^0=2^0=3^0=4^0=\cdots=1$$

Finally, point out that exponents with negative bases can be tricky:

$$-3^2=-3\times 3=-9$$

$$(-3)^2=(-3)\times(-3)=9$$

Now, go around the room asking each student in turn to calculate one of these powers:

$2^0 =$	$3^0 =$	$4^0 =$	$5^0 =$
$2^1 =$	$3^1 =$	$4^1 =$	$5^1 =$
$2^2 =$	$3^2 =$	$4^2 =$	$5^2 =$
$2^3 =$	$3^3 =$	$4^3 =$	$5^3 =$
$2^4 =$	$3^4 =$	$4^4 =$	$5^4 =$
$2^5 =$	$2^6 =$	$2^7 =$	$2^8 =$
$2^9 =$	$2^{10} =$		

When they finish calculating 3^4, stop the class and hold up 4 fingers and say, pointing each finger one at a time, "This is 3 times 3, times 3, times 3." Then clamp two of the fingers together and say "This is 3 times 3 equals 9 and the product of the remaining 3's is also 9. But you know what 9 times 9 is!" There are many ways to work a problem correctly. Some are just smarter than others are.

You should introduce the special terminology for small powers and their association with dimension at this time

Names of Small Powers

$2^0 = 1$, 2 to the 0^{th} power

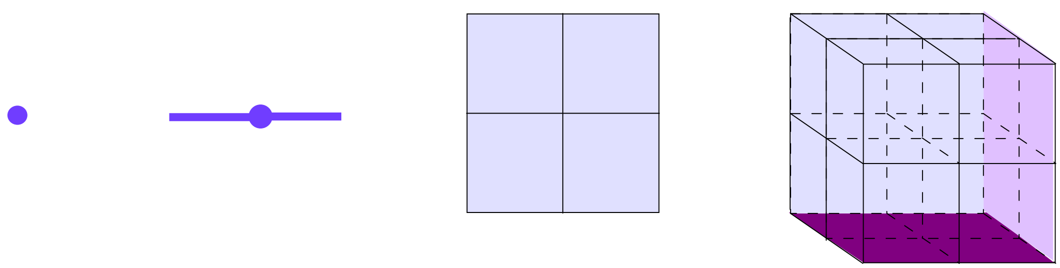
$2^1 = 2$, 2 to the 1^{st} power

$2^2 = 4$, 2 to the 2^{nd} power = 2 squared

$2^3 = 8$, 2 to the 3^{rd} power = 2 cubed

Dimensions of the Basic Geometric Objects

0 – Point	**1 – Curve**	**2 – Surface**	**3 – Solid**



size = 0	**length = 2 units**	**area = 4 units^2**	**volume = 8 units^3**

Your students may still be dubious that exponents are anything worthy of their attention. They may say, "It's good to have shorthand for repeated multiplication, but what good is repeated multiplication in the first place?" Show them the following, and have them expand a few decimal numbers[3] themselves.

$$703{,}543.069 =$$
$$7\times10^5+0\times10^4+3\times10^3+5\times10^2+4\times10^1+3\times10^0+0\times10^{-1}+6\times10^{-2}+9\times10^{-3}$$

For younger students you should just tell them that negative exponents are studied in algebra, but for now, they give a neat way of writing fractions without the fraction bar:

$$10^{-1}=\frac{1}{10^1}=\frac{1}{10},\quad 10^{-2}=\frac{1}{10^2}=\frac{1}{100},\quad 10^{-3}=\frac{1}{10^3}=\frac{1}{1000}.$$

7. Proper Punctuation & Order of Operations

We will sprinkle comments about correct punctuation throughout this section. But first, the all-important: **Never write two operation symbols in a row**. All the following expressions are wrong: $2++3$, $2+-3$, $4-+1$, $--5$, $2\div+8$, $5\times\div1$, $3\times\times5$.

Grouping Symbols

The first thing to tackle is the use of the parentheses ( ). These are called **grouping symbols**. Before you can use the expression within the parentheses you must **evaluate** that expression. All that means is you must do a side computation to calculate the value before you can combine it with some other expression. For example, in the expression $(2+3)\times4$ you must first add the *2* and *3* before multiplying by *4*.

How do you say $(2+3)$ out loud? One way is to say "parenthesis two plus three, parenthesis closed". This is a mouthful! The method we prefer is to say "the **quantity** two plus three". Thus pronounce $4\times(2+3)$ as "four times the quantity two plus three". Saying $(2+3)\times4$ is a little trickier. You should say "the quantity two plus three … times four", where the … indicates a brief pause. This way your listener understands you've reached the end of the quantity.

Here are the standard grouping symbols.

Braces: { }, Brackets: [ ], and Parentheses: ( ).

Absolute value bars: | |

Division bar: $\frac{a+b}{c+d}=(a+b)\div(c+d)$

In algebra you learn that a **function**, f(*x*), is a kind of grouping symbol. You must first determine what the value f(*x*) is before you can use it in an expression. Actually, absolute value and powers are examples of functions.

[3] Decimal numbers are not a kind of number. Decimal notation is a way of representing numbers. Likewise, fractions are another way of representing numbers. You can represent all the real numbers with decimal notation, but only the rational numbers with fractional notation. You can represent all the real numbers with something called continued fractions, but that isn't taught anymore.

PUNCTUATION:

- Parentheses, brackets, braces, and absolute value bars always come in pairs. You cannot have just one "(". You must always close it off with a ")"!
- The expression $2\times(3+4\times(5-6\times(7+8)))$ is difficult to read. This is where the brackets and braces come in.! The normal order of use is parenthesis, then brackets, then braces, but this is not an absolute. Using these we would re-write the last expression as $2\times\{3+4\times[5-6\times(7+8)]\}$. This way it is easier to see the groupings.

PRONUNCIATION:

- $|2-4|$: "the absolute value of the quantity two minus four"
- $(-2)^3$: "the quantity negative two to the third", or "the quantity negative two cubed"
- $\frac{2+3}{4+5}$ "the quantity two plus three… divided by the quantity four plus five"
- $2\times\{3+4\times[5-6\times(7+8)]\}$: "two times the quantity three plus four times the quantity five minus six times the quantity seven plus eight". Wow!

Exercises.

$2\times(5-3)=$

$(3\times 5)-10=$

$(12-7)\times(4+3)=$

$\frac{12-7}{4+3}=$

Multiplication Rule

What number does the expression $2+3\times 4$ represent? There are two ways it can be evaluated:

- Add first and then multiply. $(2+3)\times 4=5\times 4=20$
- Multiply first and then add. $2+(3\times 4)=2+12=14$

Strictly speaking we should use the parentheses for both of the above cases. Unfortunately this leads very long expressions with tons of parentheses. They can be almost unreadable. A satisfactory solution is obtained by adopting the following convention.

M**ultiplication (and division) is done before addition (and subtraction).**

Therefore, $2+3\times 4$ always evaluates to 14, and not 20.

Exercises.

$2\times 3+4\times 5-7=$

$2-3+4\times 5=$

$2-6\div 3=$

$1+2\times(1+4)\times(3-1)-6=$

Power Rule

What does $2+3^4$ mean? We could add 2 and 3 first and then take the 4th power, or we could compute 3 to the 4th and then add 3.

$$(2+3)^4 = 5^4 = 625$$

or

$$2+(3^4) = 2+81 = 83$$

. Also, what about 2×3^2? does it mean

$$(2\times 3)^2 = 6^2 = 36,$$

or

$$2\times(3^2) = 2\times 9 = 18?$$

To be utterly proper we should always use parentheses in both case, but again, we get overrun with them. To get around this we adopt another convention:

Compute powers before adding, subtracting, multiplying or dividing.

Examples.

$$-4^2 = -(4\times 4) = -16$$

$$(-4)^2 = (-4)\times(-4) = 16$$

$$1+2^3 = 1+2\times 2\times 2 = 1+8 = 9$$

$$1+(-2)^3 = 1+(-2)\times(-2)\times(-2) = 1-8 = -7$$

$$2\times(1+3\times 5^2) = 2\times(1+3\times 25) = 2\times(1+75) = 2\times 76 = 152$$

Evaluation Levels

(Do LEVEL 1 first on down to LEVEL 4)

LEVEL 1.
Evaluate sub-expressions within grouping symbols, innermost first.
LEVEL 2.
Evaluate powers, roots and functions in any order.
LEVEL 3.
Evaluate multiplications and divisions, working from left to right.
LEVEL 4.
Evaluate additions and subtractions, working from left to right.

Examples. $2+3\times 4=2+12=14$ but $(2+3)\times 4=5\times 4=20$

$2\times 3+4\times 5=6+20=26$ but $2\times(3+4\times 5)=2\times(3+20)=46$

$3\times 2^2=3\times 4=12$ but $(3\times 2)^2=6^2=36$

$2^{2^3}=2^{(2^3)}=2^8=256$ but $(2^2)^3=2^2\times 2^2\times 2^2=4^3=64$

$(-2)^4=(-2)\times(-2)\times(-2)\times(-2)=16$ but $-2^4=-16$

Finally, a real mess

$$
\begin{aligned}
3\times\{1-2^2\times\left[3+(2+\frac{2-4}{5-3})^2\right]\} &= 3\times\{1-4\times\left[3+(2+\frac{-2}{2})^2\right]\} \\
&= 3\times\{1-4\times\left[3+(2-1)^2\right]\} \\
&= 3\times\{1-4\times\left[3+1\right]\} \\
&= 3\times\{1-4\times 4\} \\
&= 3\times\{1-16\} \\
&= 3\times(-15) \\
&= -45
\end{aligned}
$$

8. Mental Math

This is a collection of helpful and fun things to learn. There are a lot more of these on the internet. The Trachtenburg method has been used to teach youngsters how do calculate long sums and products of large numbers faster than use a calculator, but this seems to me to be going overboard. There are some things, like factoring integers quickly, that are invaluable in algebra, but the main purpose I use mental math for is to give kids confidence in their ability to do math (and to learn that most problems shouldn't take more than a minute to work).

The trick is to break the problem up into easy pieces, work each piece one at a time in your head and record the results sequentially. When you have finished all the pieces, your answer is staring you in the face. (If you are lucky, you can do it in five easy pieces!)

Digit by Digit Adding and Subtracting

If there are no carries, you can add or subtract one digit at a time:

$$8375-6223=\underline{8-6},\ \underline{3-2},\ \underline{7-2},\ \underline{5-3}=2152$$
$$1375+6223=\underline{1+6},\ \underline{3+2},\ \underline{7+2},\ \underline{5+3}=7598$$

Going from left to right, or right to left, write down the answer one digit at a time.

Exercises

$$12,345+54,321=$$
$$6,092-3,051=$$
$$15.03+34.74=$$
$$1.396-0.152=$$

Tens Complement

$1+9=10$		$11+9=20$
$2+8=10$		$1.2+0.8=2.0$
$3+7=10$	so	$370+230=600$
$4+6=10$		$400+600=1000$
$5+5=10$		$20.5+5.5=26.0$

Exercises

$$1040+3060=$$
$$27.30+33.05=$$
$$99.99+00.01=$$
$$5146+5632=$$
$$280.5+420.5=$$

Using Nearby Numbers

Keep your eye out for opportunities like these.

$$9,999+7=(10,000-1)+7=10,000+6=10,006$$
$$998\times 30=(1,000-2)\times 30=1,000\times 30-2\times 30=30,000-60=29,940$$

Multiplying by Powers of 10

Multiplying by 10^n moves the decimal point *n* spaces to the <u>right</u>.

$10^0=1$	$.03\times 10^0=\frac{3}{100}\times 1=\frac{3}{100}=\underset{0}{.}03=.03$
$10^1=10$	$.03\times 10^1=\frac{3}{100}\times 10=\frac{3}{10}=\underbrace{0.}_{1}30=0.3$
$10^2=100$	$.03\times 10^2=\frac{3}{100}\times 100=3\times 1=\underbrace{03.}_{2}0=3$
$10^3=1000$	$.03\times 10^3=\frac{3}{100}\times 1000=3\times 10=\underbrace{030.}_{3}=30$

Exercises $. \; 12\times 10000 =$

$.0012\times 10^5 =$

$23.781\times 10 =$

$.123\times 10^2 =$

$10^6 =$

Multiplying by 10^{-n} moves the decimal point n spaces to the left.

$10^{-1} = 0.1 \qquad 32\times 10^{-1} = 32\times \frac{1}{10} = \frac{32}{10} = 3.\underbrace{2}_{-1} = 3.2$

$10^{-2} = .01 \qquad 32\times 10^{-2} = 32\times \frac{1}{100} = \frac{32}{100} = \underbrace{.32}_{-2} = 0.32$

$10^{-3} = .001 \qquad 32\times 10^{-3} = 32\times \frac{1}{1000} = \frac{32}{1000} = \underbrace{.032}_{-3} = .032$

$10^{-3} = .0001 \qquad 32\times 10^{-4} = 32\times \frac{1}{10000} = \frac{32}{10000} = \underbrace{.0032}_{-4} = .0032$

Exercises. $12\times .00001 =$

$.0012\times 10^{-1} =$

$23.781\times 10^{-2} = .$

$56{,}123.11\times 10^{-3} =$

$10^{-6} =$

Multiplying by 25

Look at the way this product is expanded.

$$\begin{aligned} 11\times 25 &= 25+25+25+25 \\ &+25+25+25+25 \\ &+25+25+25 \\ &=100+100+75 \\ &=275 \end{aligned}$$

Each group of four 25's adds to 100. There are 2 such groups with three 25's left over. It is easy to see that the product is 275. So how would you multiply 37 and 25? Well, every four 25's make 100, so how many fours are there in 37? There are 9 with one 25 left over. Thus $73\times 25 = 925$.

> **To multiply an integer by 25, divide the integer by 4. The quotient tells you how many 100's are in the answer and the remainder tells you how many 25's**.

Worked examples (I use "$\to$" to separate steps in a process and "$\Rightarrow$" is used in logic for the word "implies".)

$$19\times25 \to 19 = 4\times4+3 \Rightarrow \quad 19\times25 = 475$$

$$54\times25 \to 54 = 13\times4+2 \Rightarrow 19\times25 = \ 1350$$

$$-9\times25 \to \ 9 = 2\times4+1 \Rightarrow \ -9\times25 = -225$$

$$1904\times25 \to 1904 = 476\times4+0$$
$$\Rightarrow 19\times25 = 47600$$

$$(2^7+1)\times25 \to 2^7+1 = 2^5\times4+1 = 32\times4+1$$
$$\Rightarrow (2^7+1)\times25 = 3225$$

Factoring Integers

if *a*, *b* and *c* are integers, and $a = b\times c$, we say that *b* and *c* are **factors** of *a*, or that *b* and *c* **divide** *a*. It is very useful to quickly decide whether one integer divides another. The simplest divisibility rules are for 2, 5, and 10:

2 divides an integer if and only if it ends in 0, 2, 4, 6 or 8.
5 divides an integer if and only if it ends in 0 or 5.
10 divides an integer if and only if it ends in 0.

So 1,235,776 is divisible by 2, 205 is divisible by 5 and 1,290 is divisible by 10.

Slightly more complicated, but still easy to use are:

3 divides an integer if and only if the sum of its digits are divisible by 3.
9 divides an integer if and only if the sum of its digits are divisible by 9.

Therefore, 126 is divisible by 3 and also 9.

Even more complicated, but do-able is

11 divides an integer if and only if the sum of every other one of its digits minus the sum of the remaining digits is divisible by 11.

So 2,178 is divisible by 11 (and also 2 and 9) and we can quickly see (using 1 digit mental division) that

$$2{,}178 = 2\times1{,}089 = 2\times9\times121 = 2\times9\times11\times11 = 2\times3^2\times11^2$$

One Digit Mental Division

You cannot do this efficiently unless you have your multiplication tables down cold. Here is a step by step for dividing 1904 by 5 in your head:

- 5 goes into 19 3 times with a remainder of 4. Write down 3, remainder 4;
- 5 goes into 40 8 times with a remainder of 0. Write 8 next to the 3, remainder 0;
- 5 goes into 04 0 times with a remainder of 4. Write 0 next to the 8, remainder 4;
- The quotient Q=380 and the remainder R=4

Simplify a Numerical Fraction

Reduce $\frac{42 \cdot 240}{900}$. Do this one prime at a time:

- The 10's cancel so we imagine $\frac{42 \cdot 24}{90}$ in our minds. It is clear that 2, 3, 5 and 7 are primes appearing in at least one of the three factors. There may be others, but we will determine that at the end.
- $42 = 6 \times 7$, so 2 divides 42 just once and 8 divides 24 with a quotient of 3, so we have 2^4 in the numerator. 2 divides 90 with quotient 45, so we have 2^1 in the denominator. This simplifies to 2^3 in the numerator. Write $\frac{2^3}{}$.
- 3 divides 42 just once and 24 just once, so we have 3^2 in the numerator which cancels with the 9 in the denominator. Don't write down anything.
- 5 is not a factor in the numerator and appears once in the denominator. Write $\frac{2^3}{5}$.
- 7 is all that's left in the 42 and it is not a factor of the numerator. Write $\frac{2^3 \cdot 7}{5} = \frac{56}{5}$.

Simplify an Algebraic Fraction

This is actually easier than the numerical example. You can work a lot of the problem in your head while you are writing it down.

- Reduce $\frac{30 \cdot a^{3-m} \cdot b^{-2} \cdot (2 \cdot x - 1)^5}{25 \cdot a^{3+m} \cdot b^{-4} \cdot (2 \cdot x - 1)^{-3}}$.
- The primes factors are $2, 3, 5, a, b,$ and $(2 \cdot x - 1)$.
- 5 is the only common factor of 25 and 30. Write down $\frac{6}{5}$.
- The exponent of a is $3 - m - (3 + m) = -2 \cdot m$. Write down $\frac{6}{5 \cdot a^{2 \cdot m}}$.
- The exponent of b is $-2 - (-4) = 2$. Write down $\frac{6 \cdot b^2}{5 \cdot a^{2 \cdot m}}$.
- The exponent of $2 \cdot x - 1$ is $5 - (-3) = 8$. Write down $\frac{6 \cdot b^2 \cdot (2 \cdot x - 1)^8}{5 \cdot a^{2 \cdot m}}$.

Solve a Linear Equation

$$
\begin{aligned}
& 2(2p-3)=3(p+7) \\
\Rightarrow\quad & 4p-6=3p+21 \\
\Rightarrow\quad & 4p-3p=21+6 \\
\Rightarrow\quad & p=27
\end{aligned}
$$

Mental Math: The coefficient of *p* on the left hand side (lhs) is 4 and on the rhs is 3. Subtracting 3p from both sides gives p with a coefficient of 1 on the lhs. This is the denominator of the solution. The constant term on the lhs is –6 and subtracting this from both sides adds 6 to the 21 to get 27 a

9 Equations and Inequalities

If you have not used this approach to, you should try it out. I was able to get gifted fourth graders solving $3\cdot z+2=z-7$ in a few days time!

Syntax

Simple equations and inequalities are sentences whose syntax is

noun/verb/noun

lhs/verb/rhs

where the nouns are arithmetic or algebraic expressions and the verbs are

"is equal to", "is less than", "is greater than", "is less than or equal to", "is greater than or equal to", or "is not equal to".

We normally use the symbols $=,\ \neq, <, \leq, >,$ and $\geq$ as shorthand for the verbs. We use **lhs** and **rhs** as abbreviations for left hand side and right hand side. Thus, equations look like

$$x+3=5,\qquad x^2-2\cdot x+3=x+1,\qquad \frac{(x-1)\cdot(x-2)}{x+4}=0,$$

and inequalities look like

$$x+3>5,\qquad x^2-2\cdot x+3\leq x+1,\qquad \frac{(x-1)\cdot(x-2)}{x+4}\neq 0.$$

An Aside

Much of mathematics deals with simple sentences such as these. For example, in set theory the nouns are sets (and set variables) and combinations of these into set expressions. Expressions are combined using the basic operations of **intersection** ($\cap$), **union** ($\cup$) and **complement** ($\overline{\ \ }$) and look like

$$A,\ \overline{A},\ A\cup B,\ \mathrm{A}\cup B, A\cap(B\cup C),\ \text{and } (A\cup B)\cap\overline{(C\cap D)}.$$

The basic verb is "is a member of" ($\in$). This is used to define other verbs such as "is a subset of" ($\subseteq$), "is equal to" ($=$), "is not a member of" ($\notin$), "is a proper subset of" ($\subset$), and "is not equal to" ($\neq$). Thus, set equations and inequalities look like

$$x\in A,\ \ x\notin D\cap F,$$

$$A\cup X=B,\ A\cap(\overline{B}\cup X)\subset C\cap\overline{D},$$

And so on. I'm not trying to teach you set theory, but look how similar these expressions are to numerical equations and inequalities. In fact, there are amazing similarities between elementary set theory and algebra. I do want you to know what are the accepted standard notations for the different number sets (among the mathematicians I know):

The natural numbers:	$\mathbb{N} = \{0, 1, 2, ...\}$
The integers:	$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
The rational numbers:	$\mathbb{Q} = \left\{ \frac{m}{n}; \text{ such that } m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$
The real numbers:	$\mathbb{R} = \left\{ \pm \sum_{k=-n}^{\infty} \frac{a_k}{10^k}; \text{ where } n \in \mathbb{N} \text{ and the } a_{\text{k}} \text{ are digits} \right\}$
The complex numbers:	$\mathbb{C} = \left\{ a + b \cdot i; \text{ where } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}$

The Z comes from "zahlen", the German word for "numbers", and Q comes from the word "quotient". The definition of the real numbers probably has you scratching your head, but it is shorthand for all decimal numbers such as $-2,001,682.0001020...$, $\sqrt{2}$ and π. This shows you simultaneously the power of succinct notation (that 3.5 inches condenses a book) and why students hate mathematicians.

Semantics

Now that we know what these things look like, the question is, "What do they mean?" It's real simple! Equations and inequalities are shorthand for many statements, usually infinite. The statements are obtained by substituting all numbers in the range of the variable(s). For example, if x is an integer, $x + 3 = 2$ and $x + 3 > 2$ are shorthand for

$x+3=2$	$x+3>2$
$\vdots$	$\vdots$
$-2+3=2$	$-2+3>2$
$-1+3=2$	$-1+3>2$
$0+3=2$	$0+3>2$
$1+3=2$	$1+3>2$
$2+3=2$	$2+3>2$
$\vdots$	$\vdots$

At this point I would put the students of teams of two or three and give them six sentences of the form $x + a \sim b$, replacing ~ by each of $=, \neq, <, \leq, >,$ and $\geq$. Using the solution to the equality (b-a) as a center value, have them expand each equation/inequality to five or seven statements. Do this for both a and b positive, both negative, one positive and one negative, and one when both a and b are larger than 1,000,000. Have them save these for later use.

True or False

In logic, a statement is a sentence that is either true or false. Some sentences are <u>not</u> statements.

Let's eat out.	(Proposal)
Fantastic!	(Exclamation)
You need some glasses	(Suggestion)
Where is the TV remote?	(Question)
Go to bed.	(Command)

Any equation that has no variables is a statement since both its left and right hand sides evaluate to a number, so the resulting sentence must be either true or false. If there is a variable, as in $x+3=2$ or $x+3>2$, then it is neither true nor false. The question is, for what numbers is it true and for which is it false? By expanding these sentences, it is easy to determine this on a case-by-case basis.

$x+3=2$	?	$x+3>2$	?
$\vdots$	F	$\vdots$	F
$-2+3=2$	F	$-2+3>2$	F
$-1+3=2$	T	$-1+3>2$	F
$0+3=2$	F	$0+3>2$	T
$1+3=2$	F	$1+3>2$	T
$2+3=2$	F	$2+3>2$	T
$\vdots$	F	$\vdots$	T

Definition: A **solution** to an equation or inequality is any number that produces a true statement when substituted for the variable.

Thus, the (only) solution to $x+3=2$ is $x=-1$ and the (infinitely many) solutions to $x+3>2$ is $x>-1$.

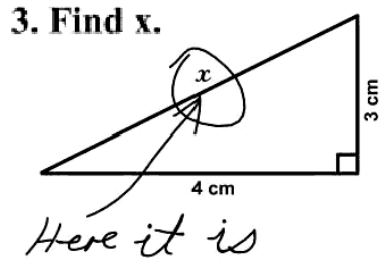
Now have your students take out those sheets where they had expanded all the $x+a\sim b$ equations and inequality, mark each statement as true or false, and write down the solutions. The intended result of this exercise is for the students to discover <u>for themselves</u> that

- In the case of the equality: Once you see that $x=-1$ is a solution, then any increase or decrease of x will lead to a corresponding increase or decrease of the lhs while the rhs remains the same, so –1 is the only solution to the problem, and
- In the case of the inequality: Once you find a value for x which is a solution, then any increase in this value will only make the lhs larger and hence, must also be a solution, and once you find a value of x which is not a solution, then any decrease in its value will make the lhs smaller and hence, will also be a non-solution

What does solve mean?

"Solve $x+3=5$" is short for "What is the number that makes $x+3=5$ true."

Be careful what you ask for!



What should a solution look like? How do I know when I'm finished?
You are done when you have an equation or inequality in the form:

variable = number, variable > number, variable < number,

variable ≥ number, variable ≤ number, or variable ≠ number,

Also acceptable is the same as above with variable and number reversed.

Now have them solve on their own some simple problems from scratch. Here are some examples:

$$x-4=2$$
$$x-4\leq 2$$
$$2x=4$$
$$2x>4$$

Finally, give them the equation $x+5,230,449=-10,546,817$, and ask them to expand it to find the solution(s) without giving them a starting point. Let them flounder around with this for a while until they start understanding the magnitude of the problem, if all you can do is guess at the answer. Point out to them that while the expanding an equation is very helpful in understanding what an equation is and what a solution is, it does not provide a practical method for finding a solution. The next few sections are all about finding such a method.

> This situation occurs over an over in mathematics. A theoretical definition is intuitively clear, but computationally ineffective, leading to a, sometimes long and difficult, search for an efficient algorithm. Theoretical definitions are most important in applied mathematics. It is usually easy to intuitively model a problem in physics with algebraic and differential equations (and a lot of other stuff) using them.

The Algebraic Method

The objective of the next few sections is to try to have the students discover the method themselves. As soon as they have discovered that when you add the same number to both sides of an equation you get an equivalent equation, challenge them to use this fact to find a sure fire method of solving equations of the form $x+a=b$.

Definition: Equivalent equations and inequalities have exactly the same solutions.

Put your students in groups of two or three and put an equation, say $x+3=2$, on the board. Have them add the same number, say 2, and subtract the same number, say 4, from both sides of the equation to get two new equations, $x+5=4$ and $x-1=-2$. then have them expand the equations and mark the statements true or false.

equation		add 2		add -4	
$x+3=2$		$x+5=4$		$x-1=-2$	
$\vdots$	F	$\vdots$	F	$\vdots$	F
$-2+3=2$	F	$-2+5=4$	F	$-2-1=-2$	F
$-1+3=2$	T	$-1+5=4$	T	$-1-1=-2$	T
$0+3=2$	F	$0+5=4$	F	$0-1=-2$	F
$1+3=2$	F	$1+5=4$	F	$1-1=-2$	F
$2+3=2$	F	$2+5=4$	F	$2-1=-2$	F
$\vdots$	F	$\vdots$	F	$\vdots$	F

Now have them do the same thing for an inequality.

inequality		add 2		add -4	
$x+3\geq 2$		$x+5\geq 4$		$x-1\geq -2$	
$\vdots$	F	$\vdots$	F	$\vdots$	F
$-2+3\geq 2$	F	$-2+5\geq 4$	F	$-2-1\geq -2$	F
$-1+3\geq 2$	T	$-1+5\geq 4$	T	$-1-1\geq -2$	T
$0+3\geq 2$	T	$0+5\geq 4$	T	$0-1\geq -2$	T
$1+3\geq 2$	T	$1+5\geq 4$	T	$1-1\geq -2$	T
$2+3\geq 2$	T	$2+5\geq 4$	T	$2-1\geq -2$	T
$\vdots$	T	$\vdots$	T	$\vdots$	T

Let them try a few more exercises like these and then challenge them to find the "theorem". This will probably be near the end of the period, so sent them home to think about it, sleep on it, and think about it some more. What you are hoping for of course is that most of them will be able to come up with a good approximation to the statement:

> **Adding or subtracting the same number to both sides of an equation or inequality yields an equivalent equation.**

So what good is this? Set things up by first asking the students what $x+0$ is. Have them substitute different numbers for *x*. Once they understand $x+0=x$, give them a problem like $u+97=124$ and ask what they would do to solve it in less than 30 seconds! In a bit remind them that a solution should look like $u=number$, so how can they get *u* by itself on the lhs? Then, what do you have to add to 97 to get 0? Let the groups work it out. The hope is that some will figure out you can go through the following sequence of equivalent equations:

$$u+97=124 \to$$
$$u+97+(-97)=124+(-97) \to$$
$$u+0=27 \to$$
$$u=27$$

This is the additive half of the algebraic method! Here are some problems for them to solve in their heads using the mental math tricks they have learned.

$$x-5=7$$
$$2+y=-3$$
$$\alpha+1.47=3.69$$
$$2\cdot b+3=b+5$$
$$u-1=u+2$$

You now go through an almost identical process for multiplication. You can cover equations and inequalities at the same time, but I think it will be less confusing if you first cover equations, then inequalities and then put them back together.
Multiply and divide both sides of an equation by the same number, expand the resulting three equations and mark true or false:

equation		multiply by 3		multiply by 1/2	
$4\cdot x=8$		$12\cdot x=24$		$2\cdot x=4$	
$\vdots$	F	$\vdots$	F	$\vdots$	F
$4\cdot(-1)=8$	F	$12\cdot(-1)=24$	F	$2\cdot(-1)=4$	F
$4\cdot(0)=8$	F	$12\cdot(0)=24$	F	$2\cdot(0)=4$	F
$4\cdot(1)=8$	F	$12\cdot(1)=24$	F	$2\cdot(1)=4$	F
$4\cdot(2)=8$	T	$12\cdot(2)=24$	T	$2\cdot(2)=4$	T
$4\cdot(3)=8$	F	$12\cdot(3)=24$	F	$2\cdot(3)=4$	F
$\vdots$	F	$\vdots$	F	$\vdots$	F

Point out that if you multiplied both sides by 0 you would get $0=0$ -- not much use. Finally conclude that

> **Multiplying or dividing both sides of an equation by the same (non-zero) number yields an equivalent equation.**

Now point out that $1\cdot x=x$ and show them (or have them figure this out on their own) the following solution sequence:

$$3\cdot z=15 \to \frac{1}{3}\cdot 3\cdot z=\frac{1}{3}\cdot 15 \to 1\cdot z=5 \to z=5$$

Here are some problems for them to do in their heads:

$$
\begin{aligned}
5\cdot x &= 35\\
35\cdot s &= 35\\
3\cdot \delta &= 396\\
2\cdot b &= 2^{15}\\
\frac{2}{3}\cdot \xi &= 6
\end{aligned}
$$

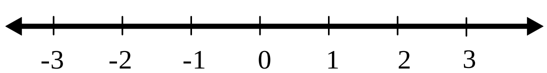
Now for inequalities: have them multiply and divide both sides of an inequality by the same number

inequality		multiply by 3		multiply by $1/2$	
$4\cdot x\le 8$		$12\cdot x\le 24$		$2\cdot x\le 4$	
$\vdots$	T	$\vdots$	T	$\vdots$	T
$4\cdot(-1)\le 8$	T	$12\cdot(-1)\le 24$	T	$2\cdot(-1)\le 4$	T
$4\cdot(0)\le 8$	T	$12\cdot(0)\le 24$	T	$2\cdot(0)\le 4$	T
$4\cdot(1)\le 8$	T	$12\cdot(1)\le 24$	T	$2\cdot(1)\le 4$	T
$4\cdot(2)\le 8$	T	$12\cdot(2)\le 24$	T	$2\cdot(2)\le 4$	T
$4\cdot(3)\le 8$	F	$12\cdot(3)\le 24$	F	$2\cdot(3)\le 4$	F
$\vdots$	F	$\vdots$	F	$\vdots$	F

Say, this looks exactly the same as the equality. We get equivalent equations! But hold on. Try multiplying and dividing by negative numbers

inequality		multiply by -3		multiply by $-1/2$	
$4\cdot x\le 8$		$-12\cdot x\le -24$		$-2\cdot x\le -4$	
$\vdots$	T	$\vdots$	F	$\vdots$	F
$4\cdot(-1)\le 8$	T	$-12\cdot(-1)\le -24$	F	$-2\cdot(-1)\le -4$	F
$4\cdot(0)\le 8$	T	$-12\cdot(0)\le -24$	F	$-2\cdot(0)\le -4$	F
$4\cdot(1)\le 8$	T	$-12\cdot(1)\le -24$	F	$-2\cdot(1)\le -4$	F
$4\cdot(2)\le 8$	T	$-12\cdot(2)\le -24$	T	$-2\cdot(2)\le -4$	T
$4\cdot(3)\le 8$	F	$-12\cdot(3)\le -24$	T	$-2\cdot(3)\le -4$	T
$\vdots$	F	$\vdots$	T	$\vdots$	T

What's going on? Remember the number line!



Multiplying by a negative number reverses the sense of an inequality

$$2<3 \Leftrightarrow -2>-3, \quad -8<-2 \Leftrightarrow 8>2, \quad -7<1 \Leftrightarrow 7>-1$$

This means we should reverse the sense of the last two inequalities and get:

inequality		multiply by -3		multiply by $-1/2$	
$4\cdot x \le 8$		$-12\cdot x \ge -24$		$-2\cdot x \ge -4$	
$\vdots$	T	$\vdots$	T	$\vdots$	T
$4\cdot(-1) \le 8$		$-12\cdot(-1) \ge -24$	T	$-2\cdot(-1) \ge -4$	T
$4\cdot(0) \le 8$	T	$-12\cdot(0) \ge -24$	T	$-2\cdot(0) \ge -4$	T
$4\cdot(1) \le 8$	T	$-12\cdot(1) \ge -24$	T	$-2\cdot(1) \ge -4$	T
$4\cdot(2) \le 8$	T	$-12\cdot(2) \ge -24$	T	$-2\cdot(2) \ge -4$	T
$4\cdot(3) \le 8$	F	$-12\cdot(3) \ge -24$	F	$-2\cdot(3) \ge -4$	F
$\vdots$	F	$\vdots$	F	$\vdots$	F

We can now state the multiplication rule for inequalities

You get an equivalent inequality if you multiply both sides by

- **A positive number**
- **A negative number and reverse the "sense" of the inequality.**

Here are some problems for your students to do in their head

$$5\cdot x < 35$$
$$-35\cdot s \ge 35$$
$$-3\cdot\delta \ne 396$$
$$2\cdot b > 2^{15}$$
$$-\frac{2}{3}\cdot\xi \le 6$$

The Algorithm

We combine what we have learned about equivalent equations and inequalities into an algorithm (a step-by-step process) that starts with the initial equation/inequality, moves through a series of equivalent equations/inequalities until an equation/inequality is reached whose solution is obvious.

Method for Equations:

1. Move all terms with variables to one side of the equation and all terms that contain only numbers to the other side by adding **negations** to both sides.
2. Multiply both sides by the **reciprocal** of the variable's <u>coefficient</u> (the number multiplying the variable)

Explaining by example is the best way to get this across
Solve $3\cdot z + 5 = 7$.

$$3 \cdot z + 5 = 7 \rightarrow$$
$$3 \cdot z + 5 + (-5) = 7 + (-5) \rightarrow$$
$$3 \cdot z + 0 = 2 \rightarrow$$
$$3 \cdot z = 2 \rightarrow$$
$$\frac{1}{3} \cdot 3 \cdot z = \frac{1}{3} \cdot 2 \rightarrow$$
$$1 \cdot z = \frac{2}{3} \rightarrow$$
$$z = \frac{2}{3}$$

Solve $5 \cdot a - 3 = 2 \cdot a - 4$

$$5 \cdot a - 3 = 2 \cdot a - 4 \rightarrow$$
$$5 \cdot a - 3 + 3 + (-2 \cdot a) = 2 \cdot a - 4 + 3 + (-2 \cdot a) \rightarrow$$
$$2 \cdot a - 0 = 0 - 1 \rightarrow$$
$$2 \cdot a = -1 \rightarrow$$
$$\frac{1}{2} \cdot 2 \cdot a = \frac{1}{2} \cdot (-1) \rightarrow$$
$$1 \cdot a = \frac{1}{2} \rightarrow$$
$$a = \frac{1}{2}$$

If the students have a problem with $5 \cdot a - 3 \cdot a = 2 \cdot a$ because they don't yet understand the distributive law, remind them that multiplication by an integer means: repeated addition. It is obvious that if you subtract the sum of 3 *a*'s from the sum of 5 *a*'s, you will get the sum of 2 *a*'s.

Method for Inequalities:

1. Move all terms with variables to one side of the equation and all terms that contain only numbers to the other side by adding **negations** to both sides.
2. Multiply both sides by the **reciprocal** of the variable's coefficient (the number multiplying the variable). **Reverse the sense of the inequality if the multiplier is negative.**

Here are some examples.
Solve. $3 \cdot z + 5 < 11$

$$3\cdot z+5<11\rightarrow$$
$$3\cdot z+5+(-5)<11+(-5)\rightarrow$$
$$3\cdot z+0<6\rightarrow$$
$$3\cdot z<6\rightarrow$$
$$\frac{1}{3}\cdot 3\cdot z<\frac{1}{3}\cdot 6\rightarrow$$
$$1\cdot z<2\rightarrow$$
$$z<2\rightarrow$$

Solve. $4\cdot z+5<z+11$

$$4\cdot z+5<z+11\rightarrow$$
$$4\cdot z+5+(-5)+(-z)<z+11+(-5)+(-z)\rightarrow$$
$$3\cdot z+0<0+6\rightarrow$$
$$3\cdot z<6\rightarrow$$
$$\frac{1}{3}\cdot 3\cdot z<\frac{1}{3}\cdot 6\rightarrow$$
$$1\cdot z<2\rightarrow$$
$$z<2$$

Here are some sample problems for your students to try.

$5\cdot x-10=35$	$4\cdot r-3=r-8$	$\frac{2}{5}\cdot t+7=9$
$5\cdot x-10\geq 35$	$4\cdot r-3<r-8$	$(-\frac{2}{5}\cdot t)+7>9$

10. Solving Factored Polynomial Equations and Inequalities (coming soon)

Inequalities solved using the easy sign-graph method.

11. Solving Factored Rational Equations and Inequalities (coming soon)

Coming soon. Inequalities solved using the easy sign-graph method.

12. Steps as Vectors (coming soon)

Coming soon. Fractions, Real Numbers, Complex Numbers, Ordered Pairs.

13. Vector Geometry (coming soon)

Coming soon. Plane and solid geometry describe by coordinate free vector algebra

Appendix. A Brief History of Math Symbols

The basic math symbols we all know were developed over many centuries. Here is a list of the basic symbols and their print historical status.

General Reference: Florian Cajori, *A History of Mathematics* (1894, 2nd ed. 1919).

Numbers

0, 1, 2, …	"zero", "one", "two", … The Indian culture developed the decimal system beginning some 5000 years ago and developed it into a rigorous numbering system, including the use of zero around 1500 years ago. The digits we use for the decimal system are the Arabic-Indian digits of 0 through 9 first codified by Abu Arrayhan Muhammad ibn Ahmad al-Biruni (973-1048).
1,000,000	Separating a numeral into groups of three with commas Charles Hutton, *Mathematical and Philosophical Dictionary*, 1795.
17.093	The decimal symbol Simon Stevin(1546~1620).
x, y, …	"a number", "the number" Using letters of the alphabet as variables, Francois Vieta (1540-1603), father of modern algebra, French. Using the fore part of the alphabet (a, b, c,…) as "the known quantity" and the hind part (u, v, w, …) as the "unknown quantity", Rene Descartes (1596-1650), French.

Operations

+	"plus" or "increased by". Originally a small Maltese cross ✠. Leonardo da Vinci (1452-1519).
−	"minus" or "decreased by" or "less" John Widmann, the father of arithmetic, 1489.
×	"times" or "multiplied by" William Oughtred, *Clavis Mathematicae*, 1631.
÷	"divided by" Johann Heinrich Rahn, *Teutsche Algebra*, 1659.
$\sqrt{\ }$	"square root of" Christoff Rudolff (1546-1620)

Comparisons

=	"is the same number as" Robert Recorde (1510-1558. *The Whetstone of Witte*, 1557), English. He justifies using two parallel line segments "... bicause noe 2 thynges can be moare equalle"
> and <	"is greater than" and "is less than" Thomas Harriot (1560-1621)

Punctuation: Grouping Symbols

( )	Nicolo Tartaglia (1506-1557), *General trattato di numeri e misure*, 1556.
[ ]	Rafael Bombelli (1526-1573), *Algebra* from about 1550.
{ }	Vieta (1540-1603), the 1593 edition of *Zetetica*